1.0

1.1

1.25   1.4   1.6

2.8   2.5
3.2   2.2
3.6
4.0   2.0

1.8

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June, 1982

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A SYSTEM FAILURE DETECTION

METHOD -- FAILURE PROJECTION METHOD

by

Xi-Cheng Lou

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SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DETREE OF

MASTER OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June, 1982

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A SYSTEM FAILURE DETECTION METHOD

-- FAILURE PROJECTION METHOD

by

Xi-Cheng Lou

Submitted to the Department of Electrical Engineering and Computer

Science on May 5, 1982 in partial fulfillment of the requirements for

the Degree of Master of Science.

ABSTRACT

This thesis proposes a system failure detection method -- Failure

Projection Method (FPM) which provides a geometric picture of the

problem of failure detection in the presence of model uncertainties

and noise.

The concept of FPM is thoroughly studied in this thesis. In

particular, two groups of formulations have been developed. One gives

distinct geometrical interpretation while the other is based on assuming

that one has available a priori information on the system state. Within

two groups three formulations which are based on slightly different

criteria and have decreasing complexity of calculation are developed.

The simplest require only a singular value decomposition. Also two

numerical examples are given which show the relationship among these

formulations and thus provide a deeper understanding of their nature.

An algebric approach is developed for the generation of a complete

set of minimal length parity checks.

The FPM is extended to including measurement and process noise.

Again a forkulation is developed which only involves a singular value

decomposition.

Again a formulation is developed which only involves a singular value decomposition.

The FPM is demonstrated on a model of three machine power system to indicate how it can be used as a design tool in assessing system redundancy and in determining parity checks.

ACKNOWLEDGEMENTS

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CHAPTER 1

INTRODUCTION

1.1 Importance of Failure Detection

System failure detection and identification are important issues

for which a great amount of work has been done in the past (see the

survey in [1]) and many of the important issues in failure detection

have been examined. There remain, however, some important problems

to be examined in order to develop useful general procedures. One of

them is the development of a design methodology for failure detection

algorithm which is robust to model uncertainties. Up to the present

there has been only one theoretical effort that focussed on this

problem. This is the work of Chow [2] and Chow and Willsky [3]. Our

research uses this previous work as a starting point from which we

shall develop significant extensions and insights into this impor-

tant problem.

1.2 Key Point of Failure Detection

As discussed by Chow [2], the failure detection process can

be considered as consisting of two stages: residual generation and

decision making based on the residuals. The key property of the

residuals is that they should behave in very different ways for each

possible failure mode so that an accurate decision can be made. That

is, we desire each failure to have a distinct signature in the resi-

duals. The actual residuals will, of course, deviate from the signatures because of the presence of noise. Also the actual failure signatures will deviate from those that we might calculate a priori because of uncertainties in the model. The design of robust failure detection systems then can be thought of in the following way:

(1.1) Find a residual generation procedure so that the residuals will be small if there is no failure and will have a distinct pattern if there is a failure. This should be true in the presence of noise and model uncertainties.

It is this problem that we will address. We will not focus any attention on the decision-making part of failure detection, where as in Chow's work [2], we think of the robustness issue completely in terms of providing residuals from which decision making is made easier.

as mentioned before, one important issue for designing a residual generator is robustness, i.e. the residual generator must be insensitive to system modelling errors or parameter variations. In Chow's thesis [2] a relationship between sensor outputs is selected so that the resulting residual will be as small as possible under worst case conditions on parameter uncertainties when no failure has ocurred. This selection depends upon the mean value of the state and the applied inputs. Therefore, different residual signals may be optimum

for different mean valures and inputs. While this may be reasonable
in some cases, there are also other problems in which it is possible
and in fact advantageous to find a single set of residual signals
that performs satisfactorily independent of operating conditions.
our proposed research -- the failure projection method (FPM) is aimed
at this problem. In this thesis, two groups of formulations are deve-
loped to solve this problem. One of them has an appealing and impor-
tant geometrical interpretation while the other is based on more
practical assumptions so that it would be more useful in practice.
Within each group, several formulations involving calculations of
decreasing complexity and starting from slightly different criteria
are given. We develop in some detail the relationships (i.e. the
similarities and differences) among these methods so that the issues
involved in choosing the appropriate formulations for a given problem
are evident.

1.3 Goal of this Thesis

The ultimate goal of this research topics is to find a vector of
residuals g such that g is small when no failure has ocurred and has
decidedly different characteristics under each of a specified set
of failures. In this thesis we take a major step forward towards the
achievement of this goal and towards the development of a deeper
understanding of the problem of robust failure detection. We do this
by examining the problem:

(1.2) Minimize the maximum residuals of the normal (unfailed) system under model uncertainties.

Note that this problem does not take any specific failure mode into account, but rather is more in keeping with the failure detection philosophy of producing as large a set of signals as possible which are all small when no failure has ocurred. The implicit assumption here is that any observable failure mode will lead to a large value for the residual vector. This implicit desire can be made explicit by considering problem such as

(1.3) Find a residual vector which yields good performance when there is only one postulated failure mode.

Here we are given one specific failure mode to detect and must find a residual that achieves an acceptable tradeoff between detection and false alarm characteristics. A variety of criteria could be used to determine an acceptable residual design. For example, one might constrain the size of the norm of the failure residuals when the system is unfailed while maximizing the failure residuals when the system fails. Clearly one can consider more complex problems in which there are several failure modes postulated and we want the residuals to be large and distinctly different for each (assuming that we wish to distinquish among the different failure modes) while still

requiring them to be small under normal conditions.

In this thesis we focus essentially all of our attention on Problem (1.2) and in particular on the solution of this problem in the presence of model uncertainties. The frame work and insight that we develop also provide a useful starting point and the requisite machinery for considering problems such as (1.3) which incorperate specific failure mode information. We will commemt on such extensions at the end of the thesis.

In Chapter 2 we introduce the basic idea behind the Failure Projection Method which is a geometrical approach using orthogonal projections. Using these ideas we develop a first group of basic formulation   to solve Problem (1.2). Three different formulations, which are based on slightly different criteria and have decreasing complexety of calculation are developed. The appealing feature of this group of formulations is their distinct geometrical interpretation and the intuition they provide for robust failure detection. In Chapter 3, another group of formulations is proposed. This group is based on assuming that one has available a priori information on the system state. Specifically, it is assumed that one knows that the state is confined to a given ellepsoid. Typically in practice one has information about the likely ranges of values for the state variables and consequently this group of approaches may be more useful in practice than that developed in Chapter 2, which uses no such information. In Chapter

4 an example of a power system with three coupled generators is studied in order to show how the formulation obtained in Chapter 3 using singular value decomposition can be applied to a practical problem.

One of the key aspects of analytic methods of failure detection is the use of information concerning the dynamic relationship among outputs. Consequently, in our geometric approach we consider residuals generated from a window of observations over some interval of time. The "observation space" then has dimension proportional to the length of this interval, and at its simplest level failure detection can be achieved by projecting the observations onto the orthogonal complement of the subspace in which the observations should lie if there is no failure. This idea in fact is the basis for our geometric approach in the next chapter for the problem of robust failure detection. In addition to providing the motivation for our robustness analysis, this idea raises an additional question. Specifically, if we are given a linear, time-invariant system, how do we determine the required length of the window of observations and also how do we generate the projection onto the orthogonal complement just mentioned. In Chapter 5 we present a frequency domain approach to determining the minimum window size and for generating this orthogonal projection.

Finally in Chapter 6, we shall briefly discuss the significance of our results and problems which should be examined in the future, including the extension of our methodology to problem such as (1.3).

CHAPTER 2

FAILURE PROJECTION METHOD (FPM)

-- GEOMETRICAL APPROACH

2.1 Introduction.

In this chapter we introduce the basic idea of the FPM and develop three formulations which have geometrical interpretations based on the notion of the angle between subspaces. They involve caculations of decreasing complexity -- from nonlinear programming to singular value decomposition.While having appealing geometrical interpretations they also have significant limitations which will be overcome in Chapter 3.

To introduce the basic idea of the FPM, let us consider a discrete time and time invariant system

$$(2.1a) \qquad x(k+1) = Ax(k) + Bu(k)$$

$$(2.1b) \qquad y(k) = Cx(k) \qquad\qquad k = 0,1,\ldots\ldots$$

As the first step we shall focus on sensor and plant failure detection and not actuator failures. Thus without loss of generality, we shall assume $u(k)=0$, $k=0,1,\ldots\ldots$ Therefore the system equations we are dealing with can be written as

$$(2.2a) \qquad x(k+1) = Ax(k)$$

$$(2.2b) \qquad y(k) = Cx(k) \qquad\qquad k=0,1,\ldots\ldots$$

where $x(k) \in R^n$, $y(k) \in R^m$, $A \in R^{n \times n}$ and $C \in R^{m \times n}$.

Define the extended observation vector of length p to be

$$
(2.3) \qquad \bar{y}_p(k) = \begin{bmatrix} y(k) \\ y(k+1) \\ . \\ . \\ y(k+p-1) \end{bmatrix} \qquad p,k=0,1,\ldots\ldots
$$

Obviously

$$
(2.4) \qquad \bar{y}_p(k) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{bmatrix} x(k)
$$

Let $s=mp$. Then $\bar{y}_p(k) \in R^s$.

Define

$$
(2.5) \qquad \bar{C}_p = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{bmatrix} \in R^{s \times n}
$$

and the observation space (of length p) Z_p to be

$$
(2.6) \qquad Z_p = R(\bar{C}_p)
$$

where $R(\bar{C}_p)$ is the *range of* $\bar{C}_p$. Define the <u>detection space</u> (of length p) G_p to be the orthogonal complement of Z_p, i.e.

$$(2.7) \qquad Z_p \oplus G_p = R^s$$

$$Z_p \perp G_p$$

Then for any vector $g \in G_p$ we have

$$(2.8) \qquad <g, \bar{y}_p> = 0$$

Equation (2.8) (or, more precisely, the left-hand side of this equation) is called a parity check [2],[4]. We shall call g a <u>detection vector</u> (of length p).

In (2.8), if p=1, then $\bar{y}_p$ =y(k). Therefore we only make use of the redundancy among values of the measurements at a single time, as considered by Potter and Suman in [4] and Desai and Ray in [5]. If p>1, we can make use of the redundancy among measurements at different time, as discussed at some length by Chow [2] and Chow and Willsky [3].

A parametric failure is one that causes parameter changes, e.g. changes in A and/or C. An additive failure is one that creates additive components in (2.2a) and/or in (2.2b). For example,

$$(2.9) \qquad x(k+1) = Ax(k)$$

$$y(k) = Cx(k) + f(k)$$

where f(k) represents some additive failure. Either of these failures will generally result in a nonzero value for at least one parity check. Thus a set of parity checks serves as a failure detector which ideally is zero if there is no failure and is non-zero if a parametric failure or an additive failure has occurred.

Motivated by these ideas, Potter and Suman [4] proposed a new approach which makes use of a complete set of detection vectors. In this approach a matrix V whose rows are an orthogonal basis for the detection space G_p (i.e. a complete set of detection vectors) is defined. Because G_p is the orthogonal complement of Z_p

$$V\bar{y}_p = 0, \qquad\qquad \text{any } \bar{y}_p \in Z_p$$

The range of V, R(V), is called the <u>parity space</u> [4],[5]. The vectors which belong to the parity space are called the <u>parity vectors.</u>

As mentioned before, in [4] and [5] the special case p=1 is considered. Here we use the definition of extended observation vector $\bar{y}_p$ for p>1 introduced by Chow in [2].

In this chapter we will make a particular choice for V. We define V to be the orthogonal projection along Z_p onto G_p denoted by P_G. Note that P_G is an operator from R^s to R^s, while in [4] V is an operator from R^s to R^q where $\dim G_p = q$. Note also that rank $P_G = q$.

It is known that an orthogonal projection has some important properties, i.e.

(2.10) $$P_G = P_G = P_G{}'$$

(2.11) $$R(P_G) = G_p \qquad N(P_G) = Z_p$$

(2.12) $$P_G = G(G'G)^{-1}G'$$

where G is a matrix the column of which form a basis for subspace G_p. We shall see later on that the geometric and algebraic properties of the orthogonal projection enable us to pose problems involving uncertainties relatively easily. This is our reason for using it.

2.2 FPM With Uncertainties -- Problem (1.2)

To characterize the influence of uncertainties let us introduce a compact set of possible parameter values U (e.g. a subset of a space R^t for some t) and two functions A and C

$$A : \quad U \to R^{n \times n}$$

$$C : \quad U \to R^{m \times n}$$

Then we can incorparate parameter uncertainties in the system equations (2.2a) and (2.2b) by modifying them as follows:

(2.13a) $x(k+1) = A(\eta)x(k)$

(2.13b) $y(k) = C(\eta)x(k)$

$k=0,1,\ldots\ldots,\qquad \eta\epsilon\ U$

Here $A(\eta)$ and $C(\eta)$ represent the possible system matrices as η varies over U. They could have different ranks for different $\eta\epsilon U$ due to unmodeled dynamics. Similarly

(2.14) $\bar{y}_p(k) = \begin{bmatrix} C(\eta) \\ C(\eta)A(\eta) \\ \vdots \\ C(\eta)A(\eta)^{p-1} \end{bmatrix} x(k) = \bar{C}_p(\eta)x(k)$

$Z_p(\eta) = R(\bar{C}_p(\eta)),\qquad \eta\epsilon\ U$

It can be seen that as η varies over U, $Z_p(\eta)$ will vary also (in fact, in general its dimension will change as η is varied). Generally we cannot find a G_p which is orthogonal to all $Z_p(\eta)$ as we did in the previous section. that is to say we cannot find a P_G which projects all vectors in $Z_p(\eta)$ (for all η) to zero. In order to formulate Problem(1.2) precisely what we must intuitively seek is a P_G which makes the projection of vectors in $Z_p(\eta)$ as small as possible for all η . To obtain some insight into this problem let us consider a simple example where $U=\{1,2,3\}$ and $Z(1),Z(2)$ and $Z(3)$ are

one-dimensional subspace of a three-dimensional space. Then one
natural choice for the detection space G is that which minimizes the
maximum projection of unit vectors in $Z(1), Z(2)$ and $Z(3)$. That is ,
this choice minimizes the worst possible effect of parameter uncertain-
ties. The nature of this approach can be visualized as in Fug. 2.1
where s_1, s_2 and s_3 are projections of unit vector in $Z(1), Z(2)$ and
$Z(3)$ onto a candidate 2-dimensional space. Our goal is to find a G
such that the maximum vector among s_1, s_2 and s_3 is minimized.
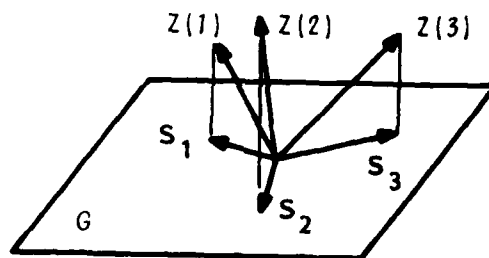
Fig. 2.1   Failure projection with system
uncertainties

Generally $Z_p(\eta)$, $\eta \varepsilon U$ may have different dimensions. Suppose

$$r = \max_{\eta \varepsilon U} \ d(Z(\eta))$$

where $d(Z(\eta))$ is the dimension of $Z(\eta)$. Then our formulation of
Problem (1.2) would be

$$(2.15) \qquad \inf_{\substack{G \\ \dim G = s-r}} \quad \sup_{\eta \in U} \quad \sup_{\substack{y \in Z(\eta) \\ \|y\| = 1}} \| P_G y \|^2$$

In the next section we shall further discuss Problem (2.15) and eventually find a way to solve it in terms of a basis of $Z(\eta)$. Furthermore, in siction 2.4 and 2.5 we shall develop other alternative formulations to problem (1./) whict turn out to be much easier to solve and also possess advan... in practice over the approach discussed in this and the following section.

2.3 Solution to Problem (2.15)

In this section we shall continue the discussion started in section 2.2 in order to understand more thoroughly the problem we have described and to develop a solution in terms of a basis of $Z(\eta)$.

Consider Eq.(2.15). Let us suppose U is a finite set, e.g.

$$(2.16) \qquad U = \{1, 2, \ldots\ldots, t\}$$

We make this assumption for mathematical convenience. However, in principle one can discretize the set of possible parameters. Furthermore, it is an unproven conjecture that any set U can be replaced by a finite set whose corresponding observation subspaces are the "extreme points" of the original set of observation spaces as the

parameter vector ranges over U.

Under the assumption (2.16), problem (2.15) can be rewritten as

$$\inf_G \; \sup_{i \in U} \; \sup_{\substack{y \in Z_i \\ \|y\|=1}} \|P_G y\|^2 \;, \quad i=1,\ldots\ldots,t$$

Suppose $d(Z_i)=m_i$. Let Z_i and G be matrices whose columns are orthonormal bases for Z_i and G respectively. Then according to (2.12) we have

$$(2.17) \qquad\qquad P_G = G(G'G)^{-1}G' = GG'$$

and

$$\sup_{\substack{y \in Z_i \\ \|y\|=1}} \|P_G y\|^2 = \sup_{\substack{y \in Z_i \\ \|y\|=1}} \|GG'y\|^2$$

Let $y=Z_i x$, $x \in R^m$. Then

$$\sup_{\substack{y \in Z_i \\ \|y\|=1}} \|P_G y\| = \sup_{\substack{\|Z_i x\|=1 \\ x \in R^m}} \|GG'Z_i x\|$$

But

$$\|Z_i x\|^2 = x'Z_i'Z_i x = \|x\|^2$$

So

$$\sup_{\substack{y \in Z_i \\ \|y\|=1}} \| P_G y \|^2 = \sup_{\substack{\|x\|=1 \\ x \in R^m}} x' Z_i' G G' G G' Z_i x = \sup_{\substack{\|x\|=1 \\ x \in R^m}} x' Z_i' G G' Z_i x$$

$$= \sigma^2_{max}(G' Z_i) = \sigma^2_{max}(Z_i' G)$$

where $\sigma_{max}(.)$ is the maximum singular value of a matrix. The last equality holds simply because $\sigma_{max}(A) = \sigma_{max}(A')$ for any matrix A. Therefore we have

(2.18)
$$\sup_{\substack{y \in Z_i \\ \|y\|=1}} \| P_G y \|^2 = \sigma^2_{max}(G' Z_i) = \sigma^2_{max}(Z_i' G)$$

Combining (2.16) and (2.18) we have that the FPM problem posed in section 2.2 reduces to solving for a $n \times (s-r)$ matrix G with orthonormal columns and which achieves

(2.19)
$$\min_{G'G=I_{s-r}} \max_i \sigma^2_{max}(G' Z_i)$$

The problem given in Eq.(2.19) has a simple geometrical interpretation. If we define the angle between two sets s_1 and s_2, $s_1, s_2 \in R^n$ to be

(2.20)
$$A(s_1, s_2) = \sup_{\substack{x \in s_1 \\ \|x\|=1}} \| P_{s_2} x \|^2$$

where $P_{s_2} x$ is the orthogonal projection of x on s_2, then formulation

(2.19) can be given a geometrical explaination. To see the meaning of

definition (2.20) let us consider a 2-dimensional case. As in Fig.2.2,

s_1 and s_2 are two subspaces. The cosine of the angle between s_1 and
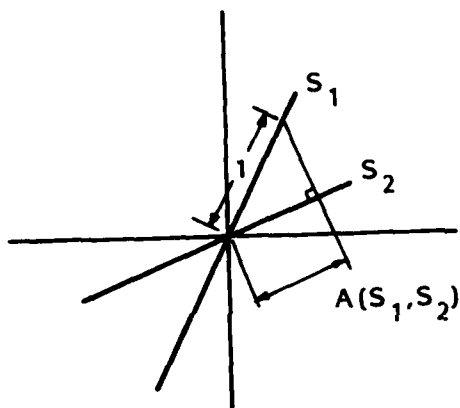
Fig. 2.2

s_2 can be seen to be the orthogonal projection of unit vector of s_1

on s_2 as shown in Fig. 2.2.

Using definition (2.20),(2.19) can be rewritten as

$$(2.21) \qquad \min_{G'G=I_{s-r}} \max_i A(Z_i,G)$$

Eg. (2.21) means that the optimal G should be that which minimizes

the maximum angle between Z_i and G. This is the geometrical meaning

of formulation (2.19).

The solution of (2.19) requires the use of nonlinear programming.

We have chosen not to do this, since as we will discuss in Chapter 3,

the mathematical formulation on which (2.19) is based has some weaknesses
in terms of its relationship to the nature of real failure detection
problems. As a prelude to the discussion in Chater 3, let us consider
several alternatives to (2.19) which are less difficult to solve,
which in some sense can be viewed as approximations to (2.19), and
which in others can be interpreted as solving slightly different
problems that possess advantages over our original formulation.

As a first step, it is easily shown that

$$\frac{\sum\limits_{i=1}^{t} \sigma_{max}^2 (G'Z_i)}{t} \leq \max_i \sigma_{max}^2 (G'Z_i) \leq \sum\limits_{i=1}^{t} \sigma_{max}^2 (G'Z_i)$$

Therefore instead of solving (2.19) one might consider solving the
following problem

$$(2.22) \qquad \min_{G'G=I} \sum\limits_{i=1}^{t} \sigma_{max}^2 (G'Z_i)$$

which gives upper and lower bounds to (2.20) because if G_1 is the
solution to (2.22) and G_2 is the solution to (2.19), then we have

$$\frac{\sum\limits_{i=1}^{t} \sigma_{max}^2 (G_1'Z_i)}{t} \leq \frac{\sum\limits_{i=1}^{t} \sigma_{max}^2 (G_2'Z_i)}{t} \leq \max_i \sigma_{max}^2 (G_2'Z_i)$$

$$\leq \max_{i} \sigma_{max}^{2}(G_1'Z_i) \leq \sum_{i=1}^{t} \sigma_{max}^{2}(G_1'Z_i)$$

or

(2.23)
$$\frac{\min_{G} \sum_{i=1}^{t} \sigma_{max}(G'Z_i)}{t} \leq \min_{G} \max_{i} \sigma_{max}^{2}(G'Z_i)$$

$$\leq \min_{G} \sum_{i=1}^{t} \sigma_{max}^{2}(G'Z_i)$$

It should be pointed out that in addition to its interpretation as an approximation to (2.19), the formulation (2.22) has its own physical interpretation. *Suppose we know the a priori probabilities of occurance of different possible uncertanties.* Then (2.19) might not be reasonable because the occurance of the uncertanty which gives maximum projection may be very unlikely. A more reasonable solution would be the minimization of the weighted summation of the maximum projections $\sigma_{max}^{2}(G'Z_i)$. That is if p_i, $i=1,....,t$ are the probabilities of the occurance of different uncertainties, then we would like to minimize the weighted summation:

(2.24)
$$\min_{G'G=I} \sum_{i=1}^{t} p_i \sigma_{max}^{2}(G'Z_i), \qquad \sum_{i=1}^{t} p_i=1$$

Geometrically here we minimize the expectation of maximum projection instead of minimizing the maximum among them.

If the uncertainties are equally likely to occur, i.e. $p_1 = p_2 = \ldots\ldots = p_t$ then we have (2.22). That is to say (2.22) is only a special case of (2.24). On the other hand, if we define new matrices $\bar{z}_i$ to be

$$\bar{z}_i = \sqrt{p_i}\, z_i \qquad\qquad i = 1,\ldots\ldots,p$$

then (2.24) can be rewitten as

$$(2.22a) \qquad\qquad \min_{G'G=I} \sum_{i=1}^{t} \sigma_{max}^{2}(G'\bar{z}_i)$$

Therefore (2.22) can be thought of as a normalized version of (2.24).

Because of inequality (2.23), it is at least plausible that in most problems the G that solves (2.22) will be a useful approximation to the G that solves (2.19). (We will provide more insight into this point through example in Section 3.3). Also because $\sum_{i=1}^{t} \sigma_{max}^{2}(G'z_i)$ is differentiable with respect to g_{ij} (the entries of G), the problem (2.22) can be solved more easily than (2.19) using nonlinear programming.

Although problem (2.19) and (2.22) can be solved in principle by nonlinear programming we would like to have an alternative approach which gives essentially a closed-form solution. Such an approach is developed in the next section, and as we will see, it also gives additional insights into the problem of failure detection in the presence of uncertain parameters.

2.4 An Alternative Approach

In this section we develop an alternative approach which can be thought of as giving an approximate solution to (2.20) and (2.22) and can be solved simply using a singular value decomposition. To begin, it is known that

$$\sigma_{max}^{2}(G'z_i) = \|G'z_i\|_2^2 = \|z_i'G\|_2^2$$

where $\|\cdot\|_2$ is the induced matrix norm. Thus (2.22) is equivalent to

$$(2.25) \qquad \min_{G'G=I} \sum_{i=1}^{t} \|z_i'G\|_2^2$$

The Frobenius norm of matrix $A\varepsilon\ C^{mxn}$ is defined to be

$$\|A\|_F^2 = \sum_i \sum_j |a_{ij}|^2$$

Because $\|A\|_F^2$ is the trace of A'A

$$\|A\|_F^2 = \sum_i \sum_j |a_{ij}|^2 = tr(A'A) = \sum_{i-1}^{n} \alpha_i$$

where α_i is the eigenvalues of A'A and

$$\|A\|_2^2 = \max_i\ \{\alpha_i\}_1^n$$

It can be easily shown that

$$\frac{\|A\|_F^2}{n} \le \|A\|_2^2 \le \|A\|_F^2$$

Therefore using the similar arguments given in proving (2.23), we have

$$(2.26) \qquad \frac{\min\limits_{G} \sum\limits_{i=1}^{t} \|Z_i'G\|_F^2}{s-r} \le \min\limits_{G} \sum\limits_{i=1}^{t} \|Z_i'G\|_2^2 \le \min\limits_{G} \sum\limits_{i=1}^{t} \|Z_i'G\|_F^2$$

Following the same line of reasoning as that used in the preceding section, we are led to the following problem.

$$(2.27) \qquad \min\limits_{G'G=I} \sum\limits_{i=1}^{t} \|Z_i'G\|_F^2$$

As mentioned before, we may use $\bar{Z}_i = \sqrt{p_i}Z_i$ instead of Z_i in (2.27) if the probabilities of occurance of uncertainties p_i are different from each other.

The problem (2.27) can further be written as

$$(2.28) \qquad \min\limits_{G'G=I} \|Z'G\|_F^2$$

where

$$(2.29) \qquad Z' = \begin{bmatrix} Z_1' \\ \vdots \\ Z_t' \end{bmatrix}$$

In the remainders of this section we discuss two very important points. The first is that (2.28) can be thought of as a different but fundamental approach to solving the problem of generating robust parity checks for failure detection. The second is that this problem can be solved very easily.

Now we derive the solution to problem (2.28). Suppose the singular values of Z are

$$(2.30) \qquad \sigma_1 \leq \sigma_2 \leq \ldots \ldots \leq \sigma_s$$

We claim that

$$(2.31) \qquad \min_{G'G=I} \|Z'G\|_F^2 = \sum_{i=1}^{s-r} \sigma_i^2$$

and the optimal G=G* is given by

$$(2.32) \qquad G^* = [\ g_1 \ldots \ldots g_{s-r}]$$

where $g_1, \ldots \ldots, g_{s-r}$ are the singular vectors of Z corresponding to $\sigma_1, \ldots \ldots, \sigma_{s-r}$ respectively.

To justify our claim we first prove the following lemma.

Lemma 2.1

If

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is symmetric and positive semidefinite, and A_{11} is m by m, A is n by n, and eigenvalures are written in increasing order, then

(2.33)
$$\lambda_i(A) \leq \lambda_i(A_{11}) , \qquad 1 \leq i \leq m$$

(2.34)
$$\sum_{i=1}^{m} \lambda_i(A) \leq \sum_{i=1}^{m} \lambda_i(A_{11}) = \text{trace}(A_{11})$$

Proof:

According to the Courant-Fischer minimax principle [10], we have

(2.35)
$$\lambda_j(A_{11}) = \min_{P_j} \max_{x \in P_j} \frac{x'A_{11}x}{x'x}$$

where P_j is a subspace of dimension j, the orthonormal basis of which is $p_1, \ldots, p_j$ Now let us define

$$q_i = \begin{bmatrix} p_i \\ 0 \end{bmatrix} , \qquad i = 1, \ldots, j$$

where p_i and q_i are m and n vectors respectively. Then (2.35) can be rewritten as

(2.36)
$$\lambda_j(A_{11}) = \min_{Q_j} \max_{y \in Q_j} \frac{y'A_{11}y}{y'y}$$

where Q_j is the subspace spanned by $q_1, \ldots, q_j$.

Also from the Courant-Fischer minimax principle,

$$(2.37) \qquad \lambda_j(A) = \min_{F} \ \max_{y \in F} \ \frac{y'Ay}{y'y}$$

where F is any subspace of dimension j. Comparing (2.36) with (2.37), it is not difficult to see that

$$\lambda_j(A) \leq \lambda_j(A_{11}) \quad , \qquad j=1,\ldots,k$$

Now let H be any matrix satisfying

$$H'H = I_s$$

$$H' = \begin{bmatrix} H_1' \\ H_2' \end{bmatrix} \qquad , \qquad H_1' \text{ is s-r by s}$$

Then

$$H'ZZ'H = \begin{bmatrix} H_1'ZZ'H_1 & * \\ * & * \end{bmatrix} = \begin{bmatrix} H_{11} & * \\ * & * \end{bmatrix}$$

and

$$\| Z'H_1 \|_F^2 = \text{trace}(H_{11})$$

According to Lemma 2.1, (2.34)

$$\sum_{i=1}^{s-r} \lambda_i(ZZ') = \sum_{i=1}^{s-r} \lambda_i(H'ZZ'H) \leq \text{trace}(H_{11}) = \| Z'H_1 \|_F^2$$

for any H satisfying $H'H = I_s$. The equality holds when the rows of H' are the eigenvectors of ZZ' corresponding to its smallest eignvalues. Since

$$\lambda_i(ZZ') = \sigma_i^2$$

we complete the proof of (2.31) and (2.32).

Therefore (2.31) and (2.32) give a very simple solution which is directly given via a singular value decomposition of Z, and thus easier to be solved than the original problem (2.19) or (2.22). It should also be pointed out that (2.31) and (2.32) have other intuitive interpretations which are given in the Appendix to this chapter.

2.5 FPM With Noise And Uncertainties

So far we have discussed the FPM with modelling uncertainties only. Introducing process and measurement noise requires some adjusment of our approach, as the observations y(k) are no longer deterministic but random vectors. In this case a good FPM must minimize both the mean value and the variance of the maximum norm of failure projection if no failures occurred. The system equations are

$$x(k+1) = A(\eta)x(k) + D(\eta)w(k)$$

(2.38)

$$y(k) = C(\eta)x(k) + v(k)$$

where $w(k)$ and $v(k)$ are driving noise and observation noise respectively and

(2.39) $E[w(k)] = 0, \qquad E[v(k)]=0, \quad k=1,2,\ldots$

(2.40) $E[w(k)w'(t)]=Q(k)\delta_{kt}$,

(2.41) $E[v(k)v'(t)]=R(k)\delta_{kt}$, $\qquad\qquad k,t=1,2,\ldots$

$$\delta_{kt} = \begin{cases} 1, & k=t \\ 0, & k\neq t \end{cases}$$

The extended observation vector $\bar{y}(k)$ can be expressed as

(2.42)
$$\bar{y}(k) = \begin{bmatrix} y(k) \\ \vdots \\ y(k+p-1) \end{bmatrix}$$

$$= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 & & \\ CD & 0 & & \\ CAD & CD & & \\ \vdots & \vdots & & 0 \\ CA^{p-2} & CA^{p-3} & CD & 0 \end{bmatrix} \begin{bmatrix} w(k) \\ \vdots \\ \vdots \\ w(k+p-1) \end{bmatrix} + \begin{bmatrix} v(k) \\ \vdots \\ \vdots \\ v(k+p-1) \end{bmatrix}$$

Define
$$\bar{\bar{C}} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{bmatrix}$$

$$\bar{D} = \begin{bmatrix} 0 & & & & \\ CD & 0 & & & \\ CAD & CD & . & & \\ \vdots & \vdots & & . & \\ CA^{p-2}D & CA^{p-3}D & & & 0 \end{bmatrix}$$

$$\bar{w}(k) = \begin{bmatrix} w(k) \\ \vdots \\ w(k+p-1) \end{bmatrix} \qquad \bar{v}(k) = \begin{bmatrix} v(k) \\ \vdots \\ v(k+p-1) \end{bmatrix}$$

Therefore

$$(2.43) \qquad \bar{y}(k) = \bar{C}x(k) + \bar{D}\bar{w}(k) + \bar{v}(k)$$

From (2.39),(2.40) and (2.41)

$$E[\bar{w}(k)] = 0 \quad , \qquad E[\tilde{v}(k)] = 0$$

$$E[\bar{w}(k)\bar{w}^{\iota}(k)] = \begin{bmatrix} Q(k) & & & \\ & . & & \\ & & . & \\ & & & Q(k) \end{bmatrix} = \bar{Q}(k)$$

$$E[\bar{v}(k)\bar{v}'(k)] = \begin{bmatrix} R(k) & & & \\ & . & & \\ & & . & \\ & & & R(k) \end{bmatrix} = \bar{R}(k)$$

Recall the deterministic case in section 2.2 where the quantity

we would like to minimize is (see (2.15))

$$(2.44) \qquad I = \max_{\substack{\eta \in U}} \max_{\substack{y \in Z(\eta) \\ \|y\|=1}} \| P_G y \|^2$$

In this present case, because the observation vector y is no longer

a deterministic vector, instead of $\| P_G y \|^2$ we would like to use

its expected value $E\|P_G y\|^2$. From (2.43) we see that

$$E\bar{y} = \bar{C}(Ex)$$

So that

$$E\bar{y} \in R(\bar{C}) = Z$$

Therefore a natural counterpart of (2.44) in stochastic case is

$$I = \max_{\substack{\eta \in U}} \max_{\substack{Ey \in Z(\eta) \\ \|Ey\|=1}} E \| P_G y(\eta) \|^2$$

Or

$$(2.45)$$

$$I = \max_{\eta} \max_{z \in Z(\eta)} E[(P_G \bar{y})'(P_G \bar{y})]$$

$$= \max_{\eta} \max_{z \in Z(\eta)} tr[E[(P_G \bar{y})(P_G \bar{y})']]$$

where

$$(2.46) \qquad z(\eta) = E(\bar{y}) = \bar{C}(\eta)x$$

Define

$$P = GG'$$

From (2.17),(2.43) and (2.46) we have

(2.47)
$$E[(P_G\bar{y}(\eta))(P_G\bar{y}(\eta))']$$

$$= F[P(z(\eta)+\bar{D}(\eta)\bar{w} + \bar{v})(z(\eta) + \bar{D}(\eta)\bar{w} + \bar{v})'P']$$

$$=E[Pz(\eta)z'(\eta)P' + P\bar{D}(\eta)\bar{Q}\bar{D}'(\eta)P' + P\bar{R}P']$$

Consider the trace of (2.47)

$$T(\eta) = tr[Pz(\eta)z'(\eta)P' + P\bar{D}(\eta)\bar{Q}\bar{D}'(\eta)P' + P\bar{R}P']$$

$$= (Pz(\eta))'(Pz(\eta)) + tr[P\bar{D}(\eta)\bar{Q}\bar{D}'(\eta)P'] + tr[P\bar{R}P']$$

$$=(Pz(\eta))'(Pz(\eta)) + tr[\bar{D}(\eta)\bar{Q}\bar{D}'(\eta)P'P] + tr[\bar{R}P'P]$$

$$=(Pz(\eta))'(Pz(\eta)) + tr[\bar{D}(\eta)\bar{Q}\bar{D}'(\eta)P] + tr[\bar{R}P]$$

Therefore (2.45) can be written as

(2.48)
$$I = \max_{\eta} \ \max_{z\epsilon Z(\eta)} \ T(\eta)$$

$$= \max_{\eta} \ \max_{z\epsilon Z(\eta)} \ \{(Pz(\eta))'(Pz(\eta)) + tr[M(\eta)P]+ tr(\bar{R}P)\}$$

$$= \max_{\eta} \ \{ \max_{z\epsilon Z(\eta)} \ [(Pz)'(Pz)] + tr[M(\eta)P] \} + tr(\bar{R}P)$$

$$= \max_{\eta} \ \{\sigma_{max}^{2}(Z'(\eta)G) + tr[M(\eta)GG'] \} + tr(\bar{R}GG')$$

where $\qquad\qquad M(\eta) = \bar{D}(\eta)\tilde{Q}\bar{\tilde{D}}'(\eta)$

Therefore Problem (1.2) becomes

$$(2.49) \qquad \min_{G'G=I} \ \max_{z \in (\eta)} \ T(\eta)$$

$$= \min_{G'G=I} \ \{\max_{\eta} \ [\ \sigma_{max}^2(Z'(\eta)G) + tr(M(\eta)GG')]+tr(\bar{R}GG')$$

If η belongs to a finite set, as we considered in Chapter 2, Section 2.3, (2.49) can be written as

$$(2.50) \qquad \min_{G'G=I} \ \{ \ \max_{i}[\sigma_{max}^2(Z_i'G)+tr(M_iGG')]+tr(\bar{R}GG')\}$$

It is obvious that (2.50) can be solved by nonlinear programming.

Using the same arguments we used in deriving (2.22) or (2.24), we arrive at an alternative criterian

$$(2.51) \qquad \min_{G'G=I} \ \{ \ \sum_{i=1}^{t} \ [\ \sigma_{max}^2(Z_i'G)+tr(M_iGG')] \ + \ tr(\bar{R}GG')\}$$

$$\min_{G'G=I} \ \{ \ \sum_{i=1}^{t} \sigma_{max}^2(Z_i'G) \ + \ tr[(\ \sum_{i=1}^{t} M_i)GG']+tr(\bar{R}GG')\}$$

Finally we have

$$(2.52) \qquad \min_{G'G=I} \left\{ \sum_{i=1}^{t} \sigma_{max}^{2}(Z_i'G) + tr(NGG') \right\}$$

where

$$N = \sum_{i=1}^{t} M_i + \bar{R}$$

It is easy to see that (2.50) and (2.52) are counterparts of (A1) and (A2) respectively. Similarly, we have the counterpart of (A3),

$$(2.53) \qquad \min_{G'G=I} \left\{ \sum_{i=1}^{t} \| Z_i'G \|_F^2 + tr(NGG') \right\}$$

Because N can be partitioned as

$$N = SS'$$

Then (2.53) can further be simplified as

$$\min_{G'G=I} \left\{ \|Z'G\|_F^2 + \|S'G\|_F^2 \right\}$$

Or

$$(2.54) \qquad \min_{G'G=I} \left\{ \|\bar{Z}'G\|_F^2 \right\}$$

where

$$\bar{Z}' = \begin{bmatrix} Z' \\ S' \end{bmatrix}$$

Therefore our problem has been reduced to the same form as (2.28) and can be solved using (2.31) and (2.32). The only difference is that instead of Z in (2.31) and (2.32), here we use $\bar{Z}=[Z\ S]$ and S is a matrix related to the covariance matrices of noise $\bar{Q}$ and $\bar{R}$. Because (2.54) can be written as

$$(2.55) \qquad \min_{G'G=I} \|\bar{Z}'G\|_F^2 = \sum_{i=1}^{s-r} \lambda_i$$

where λ_i is the eigenvalues of $\bar{Z}\bar{Z}'$ arranged in decreasing order. And

$$(2.56) \qquad \bar{Z}\bar{Z}' = ZZ' + SS'$$

Also it can be proven that [10]

$$(2.57) \qquad \lambda_i(ZZ') = \lambda_i(ZZ'+SS') \geqslant \lambda_i(ZZ')$$

Combining (2.55) '2.57) and (2.28),(2.31) we have

$$(2.58) \qquad \min_{G'G=I} \|\bar{Z}'G\|_F^2 \geq \min_{G'G=I} \|Z'G\|_F^2$$

What (2.58) tells us is that introducing noise increases the projection of the normal system. If there is no noise, S=0, then (2.55) reduces to (2.28).

The robust failure detection issue has been considered in Chow's thesis [2] where a relationship between sensor outputs is selected so that the resultant residual will be as small as possible under worst case conditions on parameter uncertainties and noise when no failure has occurred. Chow develops a minimax design procedure for determining the best residual signal. In this formulation the choice of residual signal depends upon the mean value of the state and the applied inputs. Therefore, different residual signals may be optimum for different mean values and inputs. Consequently, one can imagine a system in which different residual generation system are used under different operating conditions. In some problems, then the operating conditions are known a priori or in which conditions are such that one can consider scheduling the residual generation procedure, this approaches is clearly quite costly in terms of off-line and on-line calculations and also in many problems may be unnecessary (because there are residuals which work well under all conditions) or undesirable. The methods outlined in this chapter and in the next directly address the problem of finding a single set of residual signals that performs satisfactorily independent of operating conditions. Our formulation (2.45) can be directly interpreted as choosing the residual process under worst case conditions as opposed to under a specific condition as is done by Chow. While this formulation requires nonlinear programming, as

does Chow's, our related formulation (2.54) is far simpler to use as it only involves performing a singular value decomposition.

2.6 Summary

In this Chapter we have developed three different approaches, namely (2.19),(2.22) and (2.27) and their corresponding noise versions (2.50),(2.52) and (2.54) for the specification of robust parity checks. There are several points worth noting about these. Because of the similarity between the two groups, we focus our comments on the first group in the following discussion. The results can be easily applied to the noisy situation as well.

The first concerns numerical solution. The three problems (2.19), (2.22) and (2.27) involve calculations of decreasing complexity. Problem (2.19) and (2.22) both require nonlinear programming, but the criterion in (2.19) is non-differentiable while that in (2.22) is. Problem(2.27) involves only a singular value decomposition.

The second point is that the three criteria have intuitively appealing geometric and algebraic interpretations. In particular, (2.19) solves for the subspace that maximizes the minimum angle between the subspace and all of the possible observation spaces, while (2.22), which can be viewed as an approximation to (2.19), maximizes the expected angle between the subspace and the obser-

observation space, there the expectation is over the possible para-
meter values. Finally (2.27), which can also be viewed as an appro-
ximation to (2.22), has its own physical interpretations. For example
it can be interpreted as minimizing the volume of projection of the
Z on G (see Appendix of this Chapter).

The third point is that these formulations have significant
limitations. In particular, in computing the angle between subspaces,
one effectively examines the inner product of all possible unit
vectors in the two subspaces. An implication of this in our context
is that all directions in the observation spaces are equally likely
to occur with unity magnitude. There are two problems with this.
The first is that in the generic situation parameter variations
may increase the dimension of the observation space by introducing
a small component into the matrix $\bar{C}_p$. However in $\bar{Z}_p$ this direction is
given equal footing. Thus, the approach outlined here may produce
too few parity checks. Secondly, typically one has a significant
amount of information about the relative sizes of the state x, and
clearly this information translates into some a priori information
about the likely relative magnitudes one would expect to see in
different directions in the observation space. This information is
completely ignored in the approach described in this Chapter. In the
next Chapter we sacrifice some of the geometric interpretation to
develop three approaches that parallel (2.19), (2.22) and (2.27)
but that use the available information about x.

APPENDIX

In this Appendix we explain the geometrical meaning of formulation (2.28) in two different ways. The first one comes from looking at the problem (2.15) in the following way. First, find a subspace Z_0 of dimension r such that Z_0 is "closest" to $Z_1, \ldots, Z_t$, where "closest" means to choose a Z_0 such that

$$\min_{d(Z_0)=r} \| Z-Z_0 \|_F^2$$

here $Z=[Z_1 \ldots Z_t]$ as defined before and Z_0 is a matrix whose columns span the subspace Z_0.

Second, find the orthogonal complement of Z_0 as our G. This can be seen intuitively from Fig. A1. This G is the same as that which results from the solution of problem (2.28).
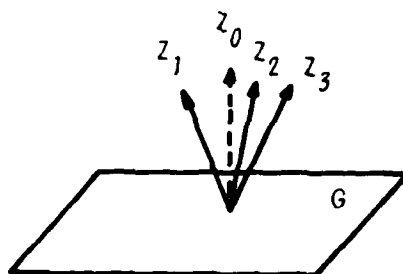
Fg. A1

To prove this result, let us first introduce the following lemma.

Lemma

Let Z be mxn real matrix. Its singular value decomposition is

$$Z = U\Sigma V$$

where

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \cdot & & & \\ & & \cdot & \cdot & 0 \\ & & & \sigma_m & \end{bmatrix} \qquad \text{(if } n \geq m)$$

$\sigma_1 \leq \ldots \leq \sigma_m$ are singular values of Z. Let r be an integer with $r \leq m$. Then the solution to the following problem

$$\begin{array}{c} \min \\ Z_0 \\ \dim(Z_0) = r \end{array} \quad \{ \|Z - Z_0\|_F^2 \}$$

is

$$Z_0 = U \begin{bmatrix} 0 & & & & & \\ & \cdot & & & & \\ & & \cdot \sigma_{m-r+1} & & 0 & \\ & & & \cdot & & \\ & & & & \cdot \sigma_m & \end{bmatrix} V$$

proof: see [10]. ◻

Let $U = [u_1 \ldots u_m]$

If $G = [u_1 \ldots u_{m-r}]$

and
$$S = \begin{bmatrix} \sigma_{m-r+1} & & \\ & \cdot & \\ & & \sigma_m \end{bmatrix}$$

Then

$$Z_0'G = V^T \begin{bmatrix} 0 & 0 \\ 0 & S \\ 0 & 0 \end{bmatrix} U^T G = V^T \begin{bmatrix} 0 & 0 \\ 0 & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = 0$$

As we see, G is the same as that of (2.28).

The other interpretation comes from the problem of choosing a G such that

$$\min_{G'G=I} \det(G'ZZ'G)$$

This problem also has an interesting interpretation. Because det(G'ZZ'G) can be thought of as the volume of the projection of the vectors in Z on G .

The solution G is the same as (2.28). The proof is similar to that of (2.28), because if we define

$$D = G'ZZ'G$$

as before, then

$$\det(D) = \prod_{i=1}^{s-r} \lambda_{di} \geq \prod_{i=1}^{s-r} \lambda_i$$

where λ_{di} and λ_i are eigenvalues of D and ZZ' respectively. The minimum arrives if

$$G = [u_1 \ldots u_{s-r}]$$

as we proved before.

CHAPTER 3

GEOMETRICAL APPROACH WITH

CONSTRAINED STATE VARIABLE NORM

3.1 Introduction

As mentioned in last Chapter, an implicit assumption we made was that all directions in the observation spaces are equally likely to occur with unity magnitude. Typically, however, this is not true in practice and would lead to undesirable designs in some situations. This can be shown by a simple example.

Consider a special case where the $\bar{C}_i$ are invertable matrices, e.g.

$$\bar{C}_i = \begin{bmatrix} 1 & 0 \\ 1+0.2i & 0.01 \end{bmatrix} \qquad , \qquad i=-1,0,+1$$

In this case $R(\bar{C}_i)=R^2$. Because $\sigma_{max}^2(Z_i{}'G)$ represent the maximum projection of unit vectors in Z_i on G, if Z_i is the whole space R^2, $\sigma_{max}^2(Z_i{}'G)$ will always be one no matter what G is chosen. Therefore (2.19) and (2.22) obviously become meaningless. Formulation (2.27) also leads to a poor choice as we will see later on.

The reason for the difficulty with (2.19) and (2.22) is that in deriving them we have effectively made the assumption that x is equally likely to lie in any direction and take on any magnitude. With this assumption $\bar{y}$ can lie in any direction in $R(\bar{C}_i)$ with no

component being more likely to have a large value than any other.

In this Chapter we shall develop several formulations which are parallel to Problem (2.19), (2.22) and (2.27) but which take into account the fact that we may have some information about the system state and thus about likely observation values. It turns out that these formulations are not only easier to solve in terms of the original system parameters, but they also give useful answers for cases such as when $\bar{C}_i$ have full rank as shown above.

3.2 Unknown But Bounded State Constraints

As mentioned in Section 3.1, in practice we always have some information about the state variable x. Typically, we at least know that the norm of x cannot be arbitrarily large. Thus we may suppose that

$$(3.1) \qquad\qquad \|x\|_Q^2 \leq M$$

where $\|x\|_Q^2 = x'Qx$, and $Q=Q'$ is a specified positive definite matrix. What (3.1) says is that x belong to an ellipsoid centered at the origin. This idea is identical to that of unknown-but-bounded variables in [6] where more detailed discussions are given in the context of problems other than the one considered here. The assumption $\|x\|_Q \leq M$ effectively allows us to include realistic statements concerning the relative magnitude of components of x (and thus of $\bar{y}$ as well).

Note that one logical choice for x is its covariance. For example, suppose x actually satisfies a linear equations

$$x_{k+1} = A_i x_k + w_k^i$$

where w_k^i is a white noise process with covariance Ξ. In this case Q_i could be thought as the solution of the Lyopunov equation

$$Q_i = A_i Q_i A_i + \Xi$$

Now, let us consider again the problem of the previous section. Because $y = \bar{C}_i x$ and $\| x \|_Q^2 \leq M$, it is natural to modify (2.17) as

(3.2)
$$\min_{G} \max_{i} \max_{\|x\|_Q^2 \leq M} \| P_G \bar{C}_i x \|^2 \quad , \quad i=1,2,\ldots,t$$

Let

(3.3)
$$Q = P'P$$

Namely P is any square-root of Q. Let G be matrix whose columns form an orthonormal basis for G. Considering (2.17) in this section, we have

$$\max_{x'Qx \leq M} \| P_G \bar{C}_i x \|^2 = \max_{x'Qx \leq M} \| GG' \bar{C}_i x \|^2$$

$$= \max_{x'P'Px \leq M} x'P'(P')^{-1} \bar{C}_i' GG' \bar{C}_i P^{-1} Px$$

$$= \max_{z'z \leq M} z'(P')^{-1}\bar{C}_i'GG'\bar{C}_iP^{-1}z$$

$$= \max_{z'z \leq 1} Mz'(P^{-1})'\bar{C}_i'GG'\bar{C}_iP^{-1}z$$

$$= M\,\sigma_{max}^2[(\bar{C}_iP^{-1})'G]$$

Thus (3.2) becomes

$$(3.4) \qquad \min_{G'G=I}\ \max_i\ \sigma_{max}^2[(\bar{C}_iP^{-1})'G]$$

which is the counterpart of problem (2.19). Because $\sigma_{max}^2[(\bar{C}_iP^{-1})'G]$ can be interpreted as the maximum projection of vectors in Z_i whose corresponding state variable x belong to an ellipsoid, (3.4) can be thought of as minimizing the maximum projections of all such vectors in $Z_1, \ldots, Z_p$. Using similar arguments we can also obtain the counterpart of Problem (2.22)

$$(3.5) \qquad \min_{G'G=I}\ \sum_{i=1}^{t} p_i\ \sigma_{max}^2[(C_iP^{-1})'G]$$

which has the same interpretation as that of (2.22). Specifically, if Q=I, i.e. the restriction on x is $\|x\|^2 \leq M$, we have

$$(3.6) \qquad \min_{G'G=I}\ \max_i\ \sigma_{max}^2(\bar{C}_i'G)$$

and

$$(3.7) \quad \min_{G'G=I} \sum_{i=1}^{t} p_i \; \sigma_{max}^2 (\bar{C}_i'G)$$

Comparing (3.4)-(3.7) and (2.19),(2.22), we see that the only difference between the two group formulations is that instead of Z_i, $\bar{C}_i P^{-1}$ is used in (3.4)and (3.5)and $\bar{C}_i$ is used in (3.6) and (3.7). As we shall see in the example at the end of this Chapter, if $\bar{C}_i P^{-1}$ is well-conditioned for all i, the formulations in this and in the previous chapter yield similar results. If this is not true, there may be a significant difference between the results. Note also that the formulations in this Chapter require only the determination of $\bar{C}_i$, which is directly related to the A_i and C_i and thus is far simpler to calculate than Z_i.

Continuing our development, we can also parallel the formulation given in eq.(2.27), which has the same advantages in terms of computational complexity -- i.e. its solution involves a singular value decomposition rather than a nonlinear optimization problem. Specifically, consider the problem

$$(3.8) \quad \min_{G'G=I} \sum_{i=1}^{t} p_i \; \|(\bar{C}_i P^{-1})'G\|_F^2$$

where the p_i are weights which can be interpreted as probabilities. Its solution is (see Eq. (2.31) and (2.32) in Section 3.2)

$$(3.9) \qquad \min_{G'G=I} \sum_{i=1}^{t} \|(\bar{C}_i P^{-1})\,'G\|_F^2 = \sum_{i=1}^{s-r} \gamma_i^2$$

where the $\bar{C}_i$ can be thought of as the normalized matrices including the weights p_i and $\gamma_1, \ldots, \gamma_{s-r}$ are the smallest $s-r$ singular values of the matrix C_a,

$$(3.10) \qquad C_a' = \begin{bmatrix} \bar{C}_1' \\ \vdots \\ \bar{C}_t' \end{bmatrix}$$

The optimal solution for G is

$$(3.11) \qquad G^* = [g_1 \ldots g_{s-r}]$$

where $g_1, \ldots, g_{s-r}$ are the left singular vectors of C_a corresponding to $\gamma_1, \ldots, \gamma_{s-r}$.

The interpretation of (3.8) is similar to that of (2.27). That is, as shown in the appendix to Chapter 2, the G which minimizes $\| G'\bar{C}_i P^{-1}\|_F^2$ subject to $G'G=I_{s-r}$ is the same G which minimizes $\det(G'\bar{C}_i P^{-2}\bar{C}_i'G)$ which has the interpretation as minimizing the volume of the projection of possible extended observation vectors on G.

As a final comment, note that $\bar{C}_i$ and Z_i can be related through

$$\bar{C}_i = Z_i T_i$$

where T_i is a full rank matrix. Therefore (3.8) can be rewritten as

(Suppose P=I)

$$(3.12) \qquad \min_{\substack{G}} \sum_{i=1}^{t} p_i \ \|(Z_i \ T_i)\,'G\|_F^2$$

If $T_i = I$ then (3.12) becomes the same as (2.27). This means that (2.27) is a special case of (3.8). In the same way we see that (3.4) and (3.5) are also the special cases of (2.19) and (2.22) respectively.

In this section we have derived three basic formulations under the assumption of constrained state variable norm. This approach allows us to take into account a priori information about the relative sizes of the state variables. Also, the dimension of the resulting parity check (i.e. of the space G) becomes a design parameter rather than being determined by the dimensions of the Z_i.

Note that from the derivation of the formulations in this Chapter we see that the results of the noise version such as (2.50), (2.52) and (2.54) can easily be extended to the results in this Chapter if we use $\bar{C}_i$ instead of Z_i in (2.50) and (2.52) and C_a instead of Z in (2.54).

In next section we will give a further, more detailed comparison between the formulations of this Chapter and those in Chapter 2. We will focus this discussion completely on the noise-free case in order to make our points more clearly.

3.3 Comparison Between the Two Groups of Formulations

So far we have obtained two groups of formulations aimed at solving problem (1.2). One set, which consists of (2.19),(2.22) and (2.27) is based on the assumption of unconstrained state variable norm, while the other which consists of (3.4), (3.5) and (3.8) is based on the assumption of constrained state variable norm. In order to facilitate further discussion, we refer to these as groups A and B and list them as follows.

$$(\text{A}_1) \qquad \min_{G'G=I} \ \max_{i} \ \sigma_{max}^2 (Z_i'G)$$

$$(\text{A2}) \qquad \min_{G'G=I} \ \sum_{i=1}^{t} \sigma_{max}^2 (Z_i'G)$$

$$(\text{A3}) \qquad \min_{G'G=I} \ \sum_{i=1}^{t} \|Z_i'G\|_F^2$$

$$(\text{B1}) \qquad \min_{G'G=I} \ \max_{i} \ \sigma_{max}^2 [(\bar{C}_i P^{-1})'G]$$

$$(\text{B2}) \qquad \min_{G'G=I} \ \sum_{i=1}^{t} \sigma_{max}^2 [(\bar{C}_i P^{-1})'G]$$

$$(\text{B3}) \qquad \min_{G'G=I} \ \sum_{i=1}^{t} \|(\bar{C}_i P^{-1})'G\|_F^2$$

The main appealing feature of group A is its geometric interpretation as discussed in Chapter 2. The basic concept behind these formulations is the angle between two subspaces, a quantity which measures the closeness between two spaces. Group B is based on more realistic assumptions, i.e. the norm of the state variable cannot be arbitrarily large, which makes it useful in some situations where group A becomes useless as we shall see in next example.

Let us consider again the example mentioned in Section 3.1. In this simple example

$$\bar{C}_1 = \begin{bmatrix} 1.0 & 0 \\ 0.8 & 0.01 \end{bmatrix} \quad \bar{C}_2 = \begin{bmatrix} 1.0 & 0 \\ 1.2 & 0.01 \end{bmatrix}$$

and by Grand-Schmitt orthogonization

$$Z_1 = \begin{bmatrix} 0.78 & -0.62 \\ 0.62 & 0.78 \end{bmatrix} \quad Z_2 = \begin{bmatrix} 0.64 & -0.77 \\ 0.77 & 0.64 \end{bmatrix}$$
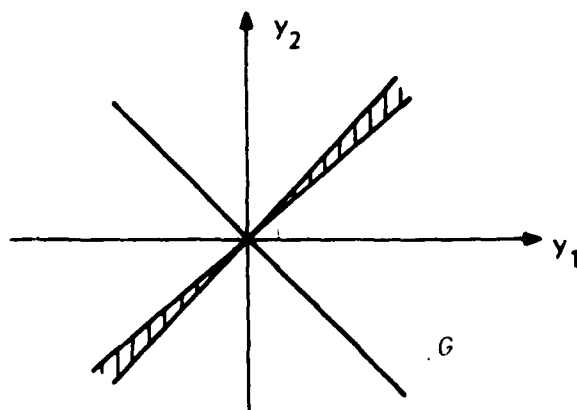
Fig. 3.1

As we point out in 3.1, (A1) and (A2) are meaningless for this example. To see that (A3) also provides a useless answer in this case but that the Group B methods yield meaningful results, consider methods (A3) and (B3) when we choose G to be a one dimensional subspace,

$$G = \begin{bmatrix} g \\ \sqrt{1-g^2} \end{bmatrix}$$

Then using formulation (A3) and (B3) we have Table 3.1

	f^*	g	$\sqrt{1-g^2}$
A3	1.99	0.993	0.121
B3	0.04	-0.714	0.699

<div align="center">Table   3.1</div>

where f^* denotes the minimum value of the criterian being minimized (in (A3) and (B3) respectively). The results are quite different and the one from B3 is the reasonable one the reason for which can be explained as follows.

In this example, if we write

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Then

$$\bar{y}(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \bar{C}_i x(k)$$

$$= \begin{bmatrix} x_1(k) \\ (1+0.2i)x_1(k) + 0.01x_2(k) \end{bmatrix}$$

If the norm of $x(k)$ is constrained to be less than M, then the effect of $x_1(k)$ on $\bar{y}(k)$ is far more important than that of $x_2(k)$. Consequently, $\bar{y}(k)$ is more likely to lie along $\begin{bmatrix} 1 \\ 1+0.2i \end{bmatrix}$ and ignoring $x_2(k)$ will not cause significant errors. This can be seen from Fig. 3.1 where the observation vector $\bar{y}$ is likely to lie in the shaded area. It is obvious the G obtained in B3 (see Table 3.1) is the reasonable one while the G resulted from A3 does not make sense. The reason for this is that the formulations in group A consider all directions in Z_i to be equally weighted and in this case this translates into assuming that $x_2(k)$ is likely to have a magnitude 100 times that of $x_1(k)$.

The previous *example illustrates* the fact that if $\bar{C}_i P^{-1}$ are very close to singular, group B will give a reasonable solution while group A will not. In the next example we will show that if $\bar{C}_i$ are well-conditioned then the results of group A and group B are quite close. In this example the system considered is scalar:

$$x(k+1) = a(n)x(k)$$
$$y(k) = cx(k)$$

where $n=1,2,3$ and $a(1)=0.8$, $a(2)=1.2$, $c=1$. Suppose $p=2$, then

$$\bar{C}_1 = \begin{bmatrix} 1.0 \\ 0.8 \end{bmatrix} \quad , \qquad \bar{C}_2 = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix} \quad , \qquad \bar{C}_3 = \begin{bmatrix} 1.0 \\ 1.2 \end{bmatrix}$$

Normalizing $\bar{C}_i$, we have

$$Z_1 = \begin{bmatrix} 0.7808 \\ 0.6247 \end{bmatrix} \quad , \quad Z_2 = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}, \qquad Z_3 = \begin{bmatrix} 0.6402 \\ 0.7682 \end{bmatrix}$$

Because in this example $s=2$, $r=1$, we see that G is a one dimensional subspace in R^2. Suppose the unit vector generating G is

$$G = \begin{bmatrix} g \\ \sqrt{1-g^2} \end{bmatrix}$$

Then

$$\sigma_{max}^2 (Z_i'G) = \lambda_{max} (G'Z_i Z_i'G)$$

$$= (z_{i1}g + z_{i2}\sqrt{1-g^2})^2$$

where z_{i1} and z_{i2} are entries of Z_i. Similarly

$$\sigma_{max}^2 (\bar{C}_i'G) = (C_{i1}g + C_{i2}\sqrt{1-g^2})^2$$

where c_{i1} and c_{i2} are entries of $\bar{C}_i$.

Because of the simple form of $\sigma_{max}^2 (\bar{C}_i'G)$ and $\sigma_{max}^2 (Z_i G)$ for this low-order example, the various problems can be solved without non-linear programming.

To solve (A1) simply observe that only Z_1, Z_3 should be taken into account. The minimum is obtained at

$$\sigma_{max}^2 (Z_1 {}^{\iota}G) = \sigma_{max}^2 (Z_2 {}^{\iota}G)$$

Formulation (A2) is simply an algebric equation for g. So that it can be solved by elementary calculus. To solve (A3) we observe that the matrix Z is

$$Z = \begin{bmatrix} z_{11} & z_{21} & z_{31} \\ z_{12} & z_{22} & z_{32} \end{bmatrix}$$

Using a singular value decomposition subroutine we can find the minimum singular value and corresponding singular vector which is the desired G in our simple case. Note that (B1), (B2) and (B3) are solved in analogous fashions.

The results are shown in table 3.2 and 3.3 where f^* is the minimum value given by each formulation

	f^*	g	$\sqrt{1-g^2}$
A1	0.01112	-0.7035	0.7106
A2	0.02026	-0.7024	0.7118
A3	0.02026	-0.7024	0.7118

Table 3.2

	f^*	g	$\sqrt{1-g^2}$
B1	0.02	-0.7071	0.7071
B2	0.0397	-0.7118	0.7024
B3	0.0397	-0.7118	0.7024

Table   3.3

From Table 3.2 and 3.3 it can be observed that the resulting detection spaces G are almost the same. This is due to the fact that the $\bar{C}_i p^{-1}$ matrices are well-conditioned.

CHAPTER   4

EXAMPLE OF FORMULATION (B3)

4.1   Introduction

In this Chapter an example of a three machine power system [11]

with realistic data is analyzed in order to show how the formulation

(B3) can be used to handle the problems in practice.

This 5th order example shows how a simple singular value decom-

position subroutine is used to solve foumulation (B3). The norm of

the maximum projection of the observation vector on G monotonely

increases with increase of dimension of G in order to keep the maximum

projection of the normal system less than a prescribed value.

Also this example illustrates how different values of p i.e.

different numbers of lagged outputs in extended observation vector,

and different C matrices influence the results.

In Section 4.2 a brief discussion of the three machine system

is given in order to explain the physical meaning of the system

parameters and the uncertainties. Section 4.3 shows the results for

different C matrices. Finally in Section 4.4 a discussion is given

to suggest how to choose the observation space in practice.

4.2 Three Machine Power System

The three machines in this power system are coupled to each other.
The whole system can be linearized as a 5th order continuous time
system [11].

(4.1)
$$\dot{x}(t) = Fx(t)$$
$$y(t) = Cx(t)$$

where

(4.2)
$$x(t) = \begin{bmatrix} \Delta\omega_r \\ \Delta\delta_c \\ \Delta\omega_c \\ \Delta\delta_d \\ \Delta\omega_d \end{bmatrix}$$

with $\Delta\omega_r, \Delta\omega_c$ and $\Delta\omega_d$ being the relative angular velocities of the
generator shafts with respect to a reference and $\Delta\delta_c$ and $\Delta\delta_d$ the relative
angles. The F matrix in (4.1) is

$$F = \begin{bmatrix} f_{11} & .00756 & .00486 & .00733 & -.00181 \\ 0 & 0 & 377 & 0 & 0 \\ .0122 & f_{32} & f_{33} & .0304 & -.00454 \\ 0 & 0 & 0 & 0 & 377 \\ -.292 & .163 & -.0292 & f_{54} & f_{55} \end{bmatrix}$$

where f_{11}, f_{33} and f_{55} are the damping factors whose values are in the
range from -.15 to -.2, and f_{32} and f_{54} are spring coefficients
whose values are not known precisely and can change from -.1 to -.4.
The constant value 377 in F is the angular frequency of the 60 Hz

current and is obviously known perfectly.

In this example we consider two C matrices.

$$(4.4) \qquad c^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$c^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Matrix c^1 incorporates the observation of $\Delta\omega_r, \Delta\omega_c$ and $\Delta\omega_d$ and c^2 includes the observation of $\Delta\omega_r, \Delta\omega_c$ and $\Delta\delta_c$.

In order to apply formulation (B3) to this example, Eq. (4.1) must be discretized as a discrete time system

$$(4.6) \qquad x(k+1) = Ax(k)$$
$$y(k) \quad = Cx(k)$$

where

$$A = \exp(F\Delta)$$

and Δ is the sampling interval.

Because the fastest angular frequency in any mode of this system is approximately 6.09 [11], we choose $\Delta=0.25s$ which is roughly 1/4 the period of the fast mode.

In next Section we apply formulation (B3) to this example and use singular value decomposition to obtain the optimum detection space corresponding to different value of p and the two C matrices.

4.3   Solution of Formulation (B3)

As discussed in Section 4.2, we assume the uncertainties only appear at eitries $f_{11}, f_{33}, f_{55}, f_{32}$ and f_{54} of F and we know the ranges of values in which these entries may lie. As mentioned in Chapter 2, Section 2.3, in order to apply (B3) to this example we must first discretize the uncertainties by assuming several "extreme points" of matrix F. Here we assume $U = \{1,2,3\}$ (see (2.16), Section 2.3) and the corresponding values $f_{11}, f_{33}, f_{55}, f_{32}$ and f_{54} may take on are shown in Table 4.1.

	f_{11}	f_{32}	f_{33}	f_{54}	f_{55}
1	-.2	-.1	-.2	-.1	-.2
2	-.15	-.4	-.15	-.4	-.15
3	-.15	-.2	-.15	-.2	-.15

Table   4.1

The next step of our procedure is to compute F_i, $A_i = \exp(F_i \Delta)$ and the corresponding $\bar{C}_i$, $i=1,2,3$.

$$(4.7) \qquad \bar{C}_i = \begin{bmatrix} C \\ CA_i \\ \vdots \\ CA_i^{\,p-1} \end{bmatrix}$$

where C can be equal to either c^1 or c^2.

In order to investigate the influence of p and C we consider

four cases,

case 1 $p=6$, $C=C^1$.

case 2 $p=6$, $C=C^2$.

case 3 $p=4$, $C=C^1$.

case 4 $p=4$, $C=C^2$.

As mentioned in Chapter 3, the solution to (B3) is

$$(4.8) \qquad \min_{G'G=I} \sum_{i=1}^{3} \| \bar{C}_i{}'G \|_F^2 = \sum_{i=1}^{s-r} r_i^2$$

where $r_1, \ldots, r_{s-r}$ are the smallest $s-r$ singular values of the matrix C_a,

$$(4.9) \qquad C_a{}' = \begin{bmatrix} \bar{C}_1{}' \\ \bar{C}_2{}' \\ \bar{C}_3{}' \end{bmatrix}$$

The optimal solution for G is

$$(4.10) \qquad G^* = [g_1 \ldots \ldots g_{s-r}]$$

where $g_1, \ldots, g_{s-r}$ are the left singular vectors of C_a corresponding to $r_1, \ldots, r_{s-r}$.

As shown in (4.8), the minimum value of $\sum_{i=1}^{3} \| \bar{C}_i{}'G \|_F^2$ monotonely increases as the dimension of G increases. In order to see the influece of the dimension of G , we define

$$(4.11) \qquad S(k) = \sum_{i=1}^{k} r_i^2$$

where r_i, i=1,......,s are the singular values of C_a ordered from smallest to largest. Eq.(4.11) gives the minimum value of $\sum_{i=1}^{3} \| \bar{C}_i \cdot C \|_c^2$ when the dimension of G equals k. Using a singular value decomposition subroutine we can compute r_i.i=1,......,s, the corresponding singular vectors and also S(k), k=1,......,s for the four different cases. The results are shown in Fig.4.1 where we have used a logarithmic scale for s(k).
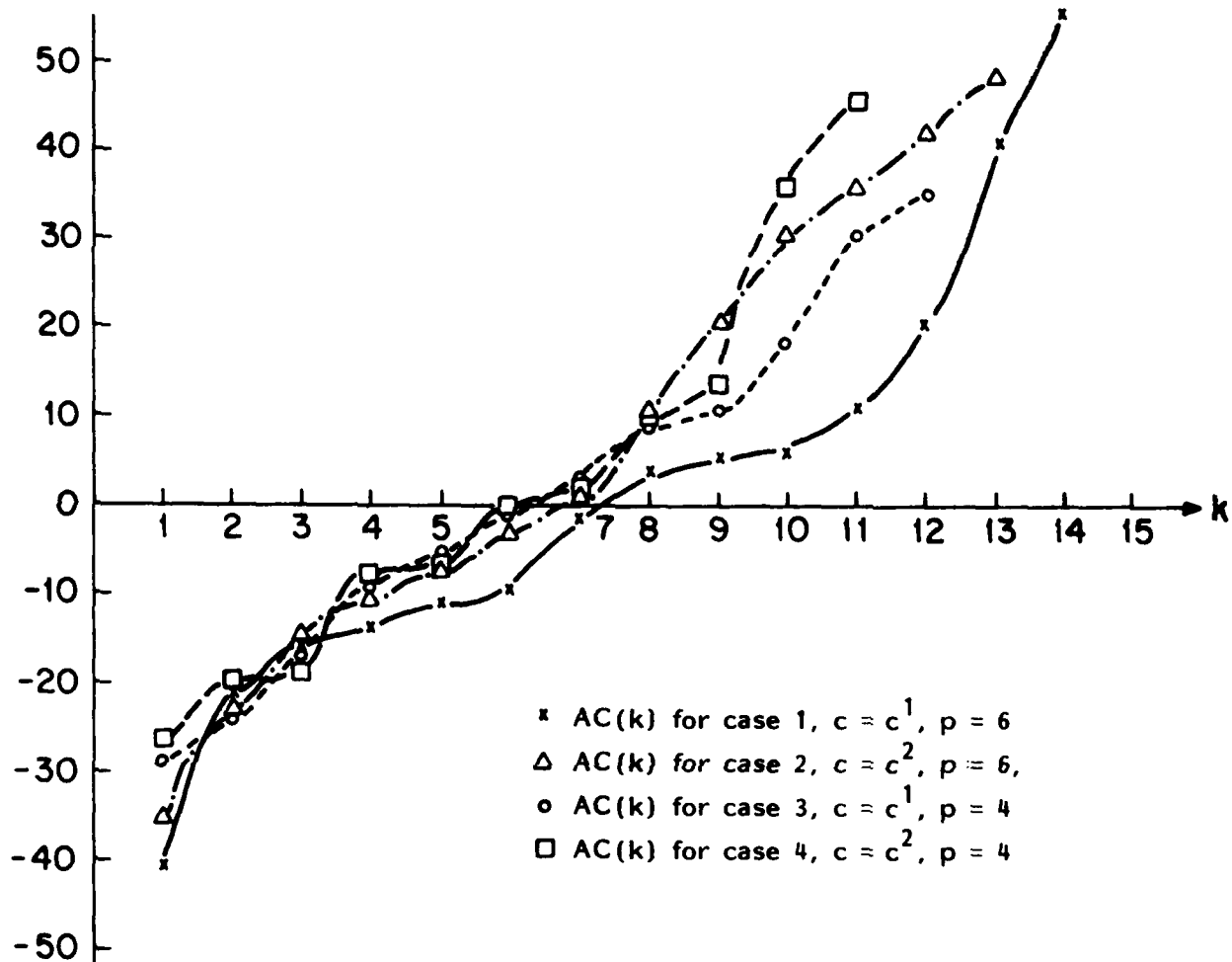
Fig. 4.1

4.4 Discussion

As we expected, $S(k) = \min_{G'G=I} \sum_{i=1}^{3} \| \bar{C}_i {}'G \|_F^2$ is a monotone function with respect to k. This can be seen from Fig. 4.1, although it is more easily seen if we use a linear scale for $S(k)$. This is done in Fig.4.2. From Fig. 4.1 and 4.2 it can be observed that in order to keep the projection of the normal system sufficiently small the dimension of the detection space should be smaller than 7 or 8 in case 2,3 and 4 and smaller than 10 in case 1, because $S(k)$ will increase very fast if k exceed these numbers.

This example illustrates how our method can be used to compare different sensor configurations (C) and different length lags (p) in the residual generation process. Also, for a given C and p, a curve as in Fig. 5.2 provides a useful visualization of the effective redundancy in the system. For example, if the curve has a dramatic "knee", i.e. a point at which the slope of the curve increases significantly (as it does for all four of our cases), one has a clear indication of how many independent parity checks can be made reliably.
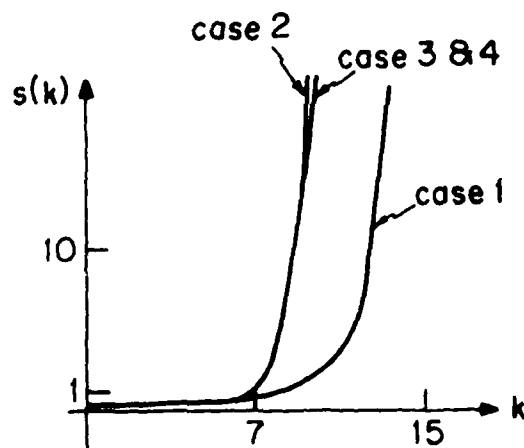
Fig. 4.2

CHAPTER 5

FREQUENCY DOMAIN DESCRIPTION OF

THE SPACE OF ALL PARITY CHECKS

5.1 Introduction

So far we have considered the failure projection method using

a geometrical approach. The principal concepts in this approach are

those of parity checks and the detection space. As defined in section

2.1, the observation space Z_p is a subspace spanned by the extended

observation vector $\bar{y}_p(k)$ where

$$(5.1) \qquad \bar{y}_p'(k)=[y'(k) \quad y'(k+1) \quad \ldots.. \quad y'(k+p-1)]'$$

Assuming there are no uncertainties, the detection space G_p is the

orthogonal complement of Z_p. We also defined a detection vector a_p

to be

$$(5.2) \qquad a_p' = [a_{p0}' \quad a_{p1}' \quad \ldots.. \quad a_{pp}']$$

which satisfies

$$(5.3) \qquad a_p'y_p(k) = 0$$

for some $p=0,1,\ldots..$ and $k=0,1,\ldots..$ (here we are using a different

symbol for detection vectors than that used in Chapter 2). Recall that

relationship (5.3) is called a parity check. It is obvious that $a_p \varepsilon \, G_p$

and finding all possible parity checks of length p is equivalent to

finding a basis for detection space G_p.

In this Chapter we shall develop a frequency domain approach

(in the case of no system uncertainty) which will answer the question

of parity checks with shortest lengths from which all possible parity

checks of all possible length can be generated. In Section 5.2 we

shall introduce a frequency domain description for parity checks by

associating a polynomial row vector with each parity check. These
polynomial detection vectors span the left null space of the

polynomial matrix [C'(ZI-A)]'. Thus, finding a polynomial basis for

this left null space is equivalent to generating all possible parity

checks. In Section 4.3 we further show that among all polynomial bases

for this null space there is a minimal basis consisting of parity

checks of minimum order. Also a method called searching the crate by

rows [7],[8] is utilized to generate this minimal basis. Thus to

generate all posssible parity checks the following procedure may be

used :

1. Generate the minimal basis $m_1(z),\ldots,m_m(z)$ by searching

   the crate by rows.

2. Form a mxm polynomial matrix M(z) whose rows are the minimal

   basis $m_1(z),\ldots,m_m(z)$.

3. Any detection vector p(z) can be generated by multiplying

   M(z) on the left by an arbitrary polynomial row vector

$$p'(z) = r'(z)M(z)$$

5.2 Frequency domain description of all possible parity checks

In this section we establish the link between the time domain
and frequency domain descriptions of the parity checks. Then we show
that the frequency domain description of all possible parity checks is
nothing more than the left null space of the polynomial matrix
$[C'(ZI-A)']'$.

Recall that the z-transform of a sequence $y(k)$, $i=0,1,\ldots$ is

$$(5.4) \qquad y(z) = \sum_{i=0}^{\infty} y(i)z^{-i}$$

Using this definition we can prove the following theorem:

__Theorem 5.1__ The following statements are equivalent

$$(5.5) \qquad 1. \quad a_p'\bar{y}_p(k) = 0 \qquad\qquad \begin{array}{l} k= 1,\ldots \quad \text{any } x_0 \varepsilon \ R^n \\ p=0,1,\ldots \end{array}$$

$$(5.6) \qquad 2. \quad p'(z)C(zI-A)^{-1} = q'(z)$$

where $q(z)$ is some nx1 polynomial vector

$$(5.7) \qquad p'(z) = \sum_{i=0}^{p} a_{p_i}'z^i$$

Proof:

$(1\to2)$: The z-Transform of (5.5) is

$$\sum_{t=0}^{p} a_{pt}'z^t(y(z) - \sum_{r=0}^{t-1} y(r)z^{-r})$$

$$= \sum_{t=0}^{p} z^t a_{pt}' \, y(z) - \sum_{t=0}^{p} z^t a_{pt}' \sum_{r=0}^{t-1} y(r) z^{-r}$$

$$= p'(z)y(z) - \sum_{t=0}^{p} a_{pt}' \sum_{r=0}^{t-1} y(r) z^{t-r} = 0$$

But

(5.8)
$$y(z) = C(zI-A)^{-1} x_0 z$$

(5.9)
$$y(r) = Cx(r) = CA^r x_0$$

Therefore the z-Transform of (5.5) is

(5.10)
$$p'(z)C(zI-A)^{-1} x_0 = \sum_{t=0}^{p} a_{pt}' \sum_{r=0}^{t-1} A^r z^{t-r-1} x_0$$

Because (5.5) is valid for all $x_0 \in R^n$, we have that

(5.11)
$$p'(z)C(zI-A)^{-1} = \sum_{t=0}^{p} a_t' C \sum_{r=0}^{t-1} A^r z^{t-r-1} = s'(z)$$

where $s(z)$ is a polynomial nx1 vector.

(2→1):   Define

(5.12)
$$g(k) = a_p' \bar{y}_p(k) = \sum_{i=0}^{p} a_{pi}' y(k+i)$$

From the preceding derivation, in particular (5.10), we know that
the z-transform of $g(k)$ is

$$g(z) = p'(z)C(zI-A)^{-1} x_0 - s'(z) x_0$$

Considering (5.6)

$$g(z) = [q'(z) \quad -s'(z)] \, x_0 \qquad \text{all } x_0 \in R^n$$

Because $q(z)$ and $s(z)$ are polynomial vectors, $g(z)$ is a scalar polynomial function. However

$$g(z) = \sum_{k=0}^{\infty} g(k) z^{-k}$$

Therefore

$$g(k) = a_p' \bar{y}_p(k) = 0 \, , \qquad k = 1,\ldots\ldots, \qquad \square$$

Since (5.5) is nothing more than a parity check and (5.7) can be thought of as its frequency domain description, this theorem sets up a link between the frequency and time domains.

Next we shall show that $p(z)$ and $q(z)$ span the left null space of the polynomial matrix $[C'(zI-A)']'$. Here the left null space of some $p \times m$ rational matrix $H(z)$ is defined [7] as a subspace spanned by $p \times 1$ rational vectors $v(z)$'s which satisfy

$$(5.13) \qquad v'(z)H(z) = 0$$

Note that (5.6) can be written as

$$p'(z)C = q'(z)(zI-A)$$

Or

$$(5.14) \qquad [p'(z) \quad -q'(z)] \begin{bmatrix} C \\ zI-A \end{bmatrix} = 0$$

Define $[p'(z) \quad -q'(z)]'$ as the polynomial detection vector and denote the left null space of $[C'(zI-A)']'$ as N_{lc}. Then from (5.14) we see that the polynomial parity vector $[p'(z) \quad -q'(z)]' \in N_{lc}$.

Based on theorem 5.1 we can also show the reverse that any polynomial vector on N_{lc} corresponds to a parity check of some length p. Therefore it is enough to know one polynomial basis of N_{lc} in order to generate all parity checks of all possible finite length.

In next section we shall discuss how to choose a specific basis consisting of elements of shortest length. We will call this a minimal basis for N_{lc}.

5.3 Minimal basis of N_{lc}

A rational vector space has a fine structure which is associated with its minimal polynomial basis [7]. The minimal basis is a basis with minimal order. Namely for the same rational vector space it is impossible to find another basis whose vectors have the same or smaller degrees than corresponding vectors in the minimal basis but with at least one vector whose degree is less than the corresponding one in the minimal basis. From an engineering point of view the minimal basis is of significance because from (5.7) and (5.5) we see that a polynomial detection vector with lower order means fewer delay elements are required. Also, intuitively, such minimal parity checks should be the least sensitive to parameter uncertainties.

To see how to generate the minimal basis of N_{lc} let us first

introduce the concept of _searching the crate by rows_ [7],[8]. The crate

is a table with m columns, representing the rows of the C matrix

$c_1',\ldots,c_m'$ and n rows corresponding to the powers $I,A,\ldots,A^{n-1}$.

So the $(i,j)^{th}$ cell of the crate represents the row vector $c_i'A^{i-1}$.

The searching process is as follows. We first search the first

row, i.e. search c_1' though c_m'. If c_j' is linearly independent of

$c_1',\ldots,c_{j-1}'$, put an x in the $(0,j)^{th}$ cell. Otherwise put a 0 in

this cell. Then search the second row, i.e. search $c_1'A$ though $c_m'A$.

If $c_j'A$ is linearly independent of $c_1',\ldots,c_m',c_1'A,\ldots,c_{j-1}'A$,

then put an x in $(1,j)^{th}$ cell. Otherwith put a 0 in that cell. We

repeat this procedure with the third row and continue in this way with

successive rows until n linearly independent vectors have been found.

c_1'	c_2'	c_3'	c_4'	
x	x	x	0	A^0
x	0	0		A^1
x				A^2
0				A^3

Fig. 5.1   Example of crate diagram

m=4, n=5

Note, that when a 0 is found then all vectors below it in the same

column will also be linearly dependent on the previous ones. For example, if

$$(5.15) \qquad c_j' = \sum_{i=1}^{j-1} a_i c_i'$$

then the vector right below it is

$$(5.16) \qquad c_j'A = \sum_{i=1}^{j-1} a_i c_i'A \qquad \text{where } a_i \text{ are some constants.}$$

but $c_i'A$, $i=1,\ldots\ldots,j-1$ are nothing more than the vectors in the second row that preceds $c_j'A$. Thus $c_j'A$ is linearly dependent on previous vectors. Furthermore, this dependence (5.16) has precisely the same form as (5.15) and clearly is derived from (5.15). Thus, (5.16) does not generate "new" polynomial parity checks. So we shall ignore (5.16) and leave the corresponding cell blank. In this way we shall have the crate diagram with m zeros and n x's (assuming that the system is observable), one example of such a diagram is shown in Fig. 5.1.

Now we indicate the role searching the crate by rows play in generating parity checks. We shall ahow that searching the crate generates m minimum length parity checks from which we can generate all other parity checks. Our procedure is as follows:

1. Show searching the crate by rows provides m parity checks.

2. These m parity checks corresponding to the minimum basis of

$$N_{1c}.$$

Step 1:

Consider the m zeros in the crate diagram (e.g. Fig. 5.1). Suppose one zero appears at the $(1,j)$ cell of the first row. From (5.15) we have

$$(5.17) \qquad [a_1 \ldots a_{j-1} \ -1 \ 0 \ \ldots 0] \begin{bmatrix} c'_1 \\ \vdots \\ c'_j \\ \vdots \\ c'_m \end{bmatrix} x(k)$$

$$= [\sum_{i=1}^{j-1} a_i c'_i - c'_j] x(k) = 0$$

Or

$$(5.18) \qquad a'y(k) = 0$$

where

$$a' = [a_1 \ldots a_{j-1} \ -1 \ 0 \ldots 0]$$

Obviously (5.18) is a parity check.

Generally, a zero in the j-th row would represent a parity check of the form

$$(5.19) \qquad [a_1' \ldots \ldots a_j'] \begin{bmatrix} y(k) \\ \vdots \\ y(k+j) \end{bmatrix} = 0$$

Its corresponding polynomial parity check would be

$$(5.20) \qquad p'(z)y(z) = q'(z)$$

where

$$(5.21) \qquad p'(z) = \sum_{i=1}^{j} a_i \, {}^{\backprime}z^{i-1}$$

In fact, any blank cell in the crate diagram is also corresponding to a linearly dependent vector and thus to a parity check as well. However, as we mentioned before, it can be generated from the m minimal ones.

<u>Step 2</u>:   Searching the crate by rows generates a minimal basis of N_{1c}.

Proof:

1. Because searching the crate by rows generates m parity checks, from Theorem 5.1 we know that they correspond to m polynomial parity vectors $[p_i'(z) \quad -q_i'(z)]$, i=1,....,m and

$$(5.24) \qquad [p_i'(z) \quad -q_i'(z)]' \; \epsilon \; N_{1c} \; , \quad i=1,2,....,m$$

and $[a_{i1} \; .... \; a_{ip}]$ is the corresponding parity check, which satisfies

$$[a_{i1}' \; .... \; a_{ip}'] \begin{bmatrix} y(k) \\ \vdots \\ \vdots \\ y(k+p-1) \end{bmatrix} = 0$$

2. From the crate diagram we can easily see, that every cell except those with x represents a parity check or equivalently a vector in the orthogonal complement of the range space of

$$\bar{C}_p = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{bmatrix}$$

In fact, the collection of parity checks corresponding to these cells span the or ogonal complements of the range spaces of $\bar{C}_p$ for all $p = 1, 2, \ldots \ldots$. In other words, they produce all possible parity checks. Therefore their corresponding parity vectors must span N_{1c}. But we have already seen that these parity vectors can be generated by the m basic ones. So the m basic polynomial parity vectors constitute the basis of N_{1c}.

3. From (5.6) we see that the order of $p(z)$ is always greater than or equal to that of $q(z)$. From the procedure of searching the crate by rows we also see that the m basic polynomial parity vectors have the minimal order.

Combine 2 and 3 we come to the conclusion that the m basic polynomial parity vectors are the basis of N_{1c} with minimal orders. Therefore they are minimal basis of N_{1c}. □

From the development in this section we have the following:

Let $m_1(z), \ldots \ldots, m_m(z)$ denote the minimal basis and let $p(z)$ denote the detection vector corresponding to any parity check. Then there is a polynomial vector $r(z)$ so that

$$(5.25) \qquad\qquad p'(z) = r'(z)M(z)$$

where
$$M(z) = \begin{bmatrix} m_1(z) \\ \vdots \\ m_m(z) \end{bmatrix}$$

5.4 Example of searching the crate

In this section we illustrate the procedure of searching the crate by rows. Specifically, consider the example specified by

$$A = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

Then
$$CA = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 0 \\ -4 & -2 & 2 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 3 & 3 & -2 \\ -2 & -1 & 1 \\ 6 & 6 & -4 \end{bmatrix}$$

Because the first two rows of C are independent of each other and the third row is dependent on the first, we have the first row of the crate diag 'm as shown in Fig. 5.2. In the same way we can determine the second and third row of the crate diagram.

c_1	c_2	c_3	
x	x	0	A^0
x	0		A^1
0			A^2

Fig. 5.2

The first parity check corresponding to the first zero in the first row is

$$a_1'C = [2 \quad 0 \quad -1]C = 0$$

with $y_1(k), y_2(k), y_3(k)$ denoting the components of the output of this system, this parity check is given by

$$2y_1(k) - y_3(k)$$

The corresponding polynomial vector $m_1(z)$ is

$$m_1(z) = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} z^0 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

Similarly, by a simple calculation it can be seen that the second parity check corresponding to the zero in the second row is

$$a_2' \begin{bmatrix} C \\ CA \end{bmatrix} = [1 \ 0 \ 0 \ 0 \ -1 \ 0] \begin{bmatrix} C \\ CA \end{bmatrix} = 0$$

This corresponds to the parity check

$$y_1(k) - y_2(k+1) = 0$$

The corresponding polynomial vector $m_2(z)$ is

$$m_2(z) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} z^0 + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} z = \begin{bmatrix} 1 \\ -z \\ 0 \end{bmatrix}$$

By simple linear algebra we can show that the third parity check corresponding to the third zero in the third row is

$$a_3' = [-1 \quad 1 \quad 0 \quad -2 \quad 0 \quad 0 \quad -1 \quad 0 \quad 0]$$

This corresponds to the parity check

$$-y_1(k) + y_2(k) - 2y_1(k+1) - y_1(k+2)$$

The corresponding polynomial vector is

$$m_3(z) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} z^0 + \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} z + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} z^2 = \begin{bmatrix} -1-2z-z^2 \\ 1 \\ 0 \end{bmatrix}$$

Then

$$M(z) = \begin{bmatrix} m_1'(z) \\ m_2'(z) \\ m_3'(z) \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & -z & 0 \\ -1-2z-z^2 & 1 & 0 \end{bmatrix}$$

and all other parity checks can be generated as in Eq. (5.25).

Conclusion

In this Chapter a link between the time domain and frequency domain descriptions of parity checks has been established. Based on this link, it has been proven, that generating all possible parity checks is equivalent to generating a basis for the left null space of the polynomial matrix $[C'(zI-A)']'$. Finally, a method called searching the crate by rows has been shown to generate the minimal basis for the left null space of the polynomial matrix $[C'(zI-A)']'$, i.e. N_{lc}.

CHAPTER   6

CONCLUSION

6.1 Main Contributions of This Thesis

The main contributions of this thesis are as follows.

1. The formulation of the Failure Projection Method which

provides a geometric picture of the problem of failure detection in

the presence of model uncertainties and noise.

2. The thorough development of the FPM concept. In particular,

two groups of formulations have been developed. One gives distinct

geometrical interpretation while the other is based on assuming that

one has available a priori information on the system state. Within

two groups three formulations which are based on slightly different

criteria and have decreasing complexity of calculation are developed.

The simplest require only a singular value decomposition. Also two

numerical examples are given which show the relationship among these

formulations and thus provide a deeper understanding of their nature.

3. An algebric approach has been developed for the generation

of a complete set of minimal length parity checks.

4. Extension of FPM to include measurement and process noise.

Again a formulation is developed which only involves a singular

value decomposition.

5. An illustration of FPM to indicate how it can be used as a

design tool in assessing system redundancy and in determining parity checks.

6.2 Further work

We feel that the following represent the most important directions for further work.

1. As mentioned in Chapter 1, in this thesis we only examined Problem (1.2) ---- minimizing the maximum residuals of the normal system under model uncertainties and noise. In the next section we briefly discuss some possible solutions for problem (1.3), i.e. finding a failure detector which yields good performance when there is only one postulated failure mode, using the machinary developed in this thesis. The completion of this step and the final solution of (1.3) and our ultimate goal Problem (1.1) are the important subjects for future work.

2. In section 2.3 we made the assumption that the set of uncertainties U is a finite set

$$U = \{1, 2, \ldots, t\}$$

and pointed out that it is a unproven conjecture that any set U can be replaced by a finite set whose corresponding observation subspaces are the "extreme points" of the original set of observation spaces as the parameter vector ranges over U. This conjecture should be proven.

3. The algebraic approach developed in Chapter 4 provides a useful starting point for further investigating the algebraic aspects of failure detection. For example, in Chapter 4 we only considered parity checks of finite length p. But in practice parity checks with infinite length are often used. A simple example of a infinite length parity checks might be as follows.

Consider a system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Therefore
$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & T \\ 0 & 1 \end{bmatrix}$$

It can be seen that a parity check would be

$$\sum_{k=0}^{\infty} [x_1(k) - x_1(k) + Tx_2(k)] = 0$$

which is of infinite length.

6. 3 FPM with a priori information about one failure mode

In the preceding chapters we focussed attention on Problem (1.2)
(see Chapter 1, Section 1.3) without taking any specific failure mode
into account. In this section we briefly consider the Problem (1.3),
where one specific failure mode is given. To solve Problem (1.3)
we must find a residual that achieves an acceptable tradeoff between
detection and false alarm characteristic. As mentioned in Section 1.3
one possible criterian for this problem is to constrain the norm of
the failure residuals when the system is unfailed while maximizing
the failure residuals when the system fails.

As we did in Chapter 2 , Section 2.2, we may consider the
worst-case situation. Namely we may constrain the maximum projection
of the extended observation onto the detection space G to be less
than a prescribed quantity when there is no failure, while maximizing
the minimum projection of the observation vector when there is a failure.
This leads to the following optimization problem.

$$\max_{G} \min_{i} \min_{\substack{y \in Z_{fi} \\ \|y\|=1}} \| P_G y \|^2$$

under constraints $G'G = I_{s-r}$

(6.1)

$$\max_{i} \max_{\substack{z \in Z_{ui} \\ \|z\|=1}} \| P_G z \|^2 \leq M$$

where Z_{ui}, $i=1,\ldots,p$ are the observation spaces of the unfailed system and Z_{fi} are the observation spaces of the failed system. Using the same arguments in Section 2.3 in deriving Eq.(2.18), we can rewrite (6.1) as

$$\max_{G} \min_{i} \; \sigma_{\min}(Z_{fi}'G)$$

(6.2)
$$G'G = I_{s-r}$$

$$\max \; \sigma_{\max}(Z_{ui}'G) \leq M$$

where $\sigma_{\min}(\cdot)$ is the minimum singular value of some matrix. Instead of considering the worst case one can imagine the possibility of using the weighted summation or expectation as condidered in section 2.3. Then we have

$$\max \; \sum_{i=1}^{p} \sigma_{\min}(\bar{Z}_{fi}'G)$$

(C2)
$$G'G = I_{s-r}$$

$$\sum_{i=1}^{p} \sigma_{\max}(\bar{Z}_{ui}'G) \leq M$$

where $\bar{Z}_{fi}$ and $\bar{Z}_{ui}$ should be considered as the normalized version of Z_{fi} and Z_{ui} as we did in Section 2.3 (see formulation (2.22a)). In principle (C1) and (C2) can be solved by nonlinear programming, although this is not an easy task.

Using criteria other than those developed so far, we can obtain other formulations. For example, if we choose G to minimize the difference between the projection norms of the failed and unfailed systems we will have a set of formulations which have the same general objective as those described previously in this section.

As one example, a formulation similar to (C2) would be

$$(D2) \qquad \max_{G'G=I} \left[\sum_{i=1}^{p} \sigma_{min}(\bar{z}_{fi}'G) - \sum_{i=1}^{p} \sigma_{max}(\bar{z}_{ui}'G) \right]$$

Also using the Frobenius norm, (D2) can be rewritten as

$$(D3) \qquad \min_{G'G=I} \left(\| \bar{z}_{u}'G \|_F^2 - \| \bar{z}_{f}'G \|_F^2 \right)$$

where

$$\bar{z}_f' = \begin{bmatrix} \bar{z}_{f1}' \\ \vdots \\ \bar{z}_{fp}' \end{bmatrix}, \qquad \bar{z}_u' = \begin{bmatrix} \bar{z}_{u1}' \\ \vdots \\ \bar{z}_{up}' \end{bmatrix}$$

Formulation (D3) can also be written as

$$(D3a) \qquad \min_{G'G=I} \; tr(G'ZSZ'G)$$

where

$$Z' = \begin{bmatrix} \bar{z}_u' \\ \bar{z}_f' \end{bmatrix}, \qquad S = \begin{bmatrix} I_p & \\ & -I_q \end{bmatrix}$$

It is easy to see that (D3a) can be solved using the same derivation we used to solve (A3) and (B3). The result is

$$\min_{G'G=I} \quad \text{tr}(G'ZSZ'G) = \sum_{i=1}^{s-r} \lambda_i$$

$$G^* = [g_1 \ldots \ldots g_{s-r}]$$

where λ_i are smallest $s-r$ eigenvalues of ZSZ' and $g_1, \ldots \ldots, g_{s-r}$ are corresponding eigenvectors.

Problem (D3a) can be solved by "S-singular value decomposition". If we denote

$$\lambda_s = \sigma_1^2, \quad \lambda_{s-1} = \sigma_2^2, \quad \ldots \ldots, \quad \lambda_{s-p+1} = \sigma_p^2, \quad \lambda_{s-p+1} = -\sigma_{p+1}^2, \ldots \ldots,$$

$$\lambda_1 = -\sigma_{p+q}^2$$

then we know that

$$ZSZ' = U \begin{bmatrix} \sigma_1^2 & & & & & \\ & \cdot & & & & \\ & & \cdot & \sigma_p^2 & & \\ & & & & -\sigma_{p+1}^2 & \\ & & & & & \cdot \\ & & & & & & -\sigma_{p+q}^2 \end{bmatrix} U'$$

$$= U\Sigma S \Sigma U'$$

where

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \cdot & \cdot & \\ & & \cdot & \\ & & & \cdot \sigma_{p+q} \\ & & & \end{bmatrix} \quad , \ U'U = I$$

If Σ is nonsingular we have

$$(\Sigma^{-1}U'Z)S(Z'U\Sigma^{-1}) = S$$

Define

$$V = \Sigma^{-1}U'Z$$

we have

$$VSV' = S$$

V is called S-orthogonal. Therefore

$$(6.3) \qquad\qquad Z = U\Sigma V$$

Then the solution G of rank r is obtained by taking last r columns

of U. The thorough examination of the problem and approach described

in this section remains as a useful direction for future research.

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